$\boxed{3.27}$ If $g(u) = u - \frac{1}{u}$, show $\frac{dg}{du}(c) = \frac{c^2+1}{c^2}$

Solution:

$$\lim_{h\to 0} \frac{g(u+h)-g(u)}{h} = \lim_{h\to 0} \frac{\left((u+h) - \frac{1}{(u+h)}\right) - \left(u - \frac{1}{u}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{u}+h - \frac{1}{u+h} - \cancel{u} + \frac{1}{u}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}_{1} + \frac{-\frac{1}{u+h} + \frac{1}{u}}{h}$$

$$= \lim_{h\to 0} 1 + \frac{\frac{-u}{u(u+h)} + \frac{u+h}{u(u+h)}}{h}$$

$$= \lim_{h\to 0} 1 + \frac{\frac{-\cancel{u}+\cancel{u}+h}{u(u+h)}}{h}$$

$$= \lim_{h\to 0} 1 + \frac{\cancel{h}}{u(u+h)} \cdot \frac{1}{\cancel{h}} = 1 + \frac{1}{u^2}$$

$$g'(u) = \frac{u^2+1}{u^2}$$

$$\Rightarrow \boxed{g'(c) = \frac{c^2+1}{c^2}}. \blacksquare$$

# More formulas for derivatives

So far: $(\sin(x))' = \cos(x)$

$(x^n)' = n x^{n-1}$

General Fact about derivatives of functions: Linearity.

- If $f$ and $g$ are differentiable (ie $f'$ and $g'$ exist) at $x$ and $c \in \mathbb{R}$ is a constant, then:

$$\to \quad (c f(x))' = c f'(x)$$

$$\to \quad ((f+g)(x))' = f'(x) + g'(x).$$

Example proof (first fact):

$$((cf)(x))' = \lim_{h \to 0} \frac{(cf)(x+h) - (cf)(x)}{h}$$

$$= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c f'(x). \qquad \square$$

by the algebraic limit thm.

**Example** Find the derivative of

$$F(x) = 14\sin(x) + \frac{(x+1)^3}{\sqrt{x}}.$$

---

Solution:

$$F(x) = 14\sin(x) + \frac{(x^3+3x^2+3x+1)}{x^{1/2}}$$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \end{array}$$

$$\Rightarrow F(x) = 14\sin(x) + \frac{x^3}{x^{1/2}} + \frac{3x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$\Rightarrow F(x) = 14\sin(x) + x^{5/2} + 3\cdot x^{3/2} + 3x^{1/2} + x^{-1/2}$$

$$\Rightarrow F'(x) = \left[14\sin(x) + x^{5/2} + 3\cdot x^{3/2} + 3x^{1/2} + x^{-1/2}\right]'$$

$$= [14\sin(x)]' + (x^{5/2})' + (3x^{3/2})' + (3x^{1/2})' + (x^{-1/2})'$$

$$= 14\cdot(\sin(x))' + \frac{5}{2}x^{3/2} + 3(x^{3/2})' + 3(x^{1/2})' + \left(-\frac{1}{2}\right)x^{-3/2}$$

$$= 14\cos(x) + \frac{5}{2}x^{3/2} + 3\cdot\left(\frac{3}{2}\right)x^{1/2} + 3\left(\frac{1}{2}\right)x^{-1/2} + \left(-\frac{1}{2}\right)x^{-3/2}$$

$$= \boxed{14\cos(x) + \frac{5}{2}x^{3/2} + \frac{9}{2}x^{1/2} + \frac{3}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}}$$

---

Note: subtraction also works

$$\left((f-g)(x)\right)' = \left(f(x) + (-1)g(x)\right)'$$

$$= f'(x) + \left((-1)g(x)\right)'$$

$$= f'(x) + (-1)g'(x) = f'(x) - g'(x).$$

## Next Shortcut rule:

$$(e^x)' = e^x.$$

Proof: $$(e^x)' = \lim_{h\to 0} \frac{e^{(x+h)} - e^{(x)}}{h}$$

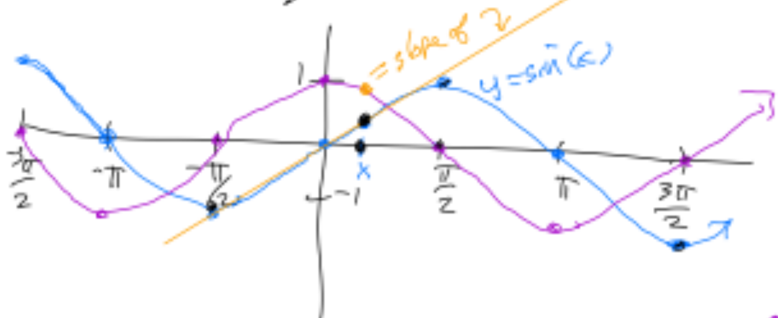
$$= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x (e^h - 1)}{h} = e^x. \quad \blacksquare$$

## Next shortcut rule: $(\cos(x))' = ??$

From before:

$$(\sin(x))' = \cos(x).$$




So: Since cos(x)'s graph is the same as sin(x) moved to the left $\frac{\pi}{2}$,

$$\cos(x) = \sin(x + \pi/2)$$

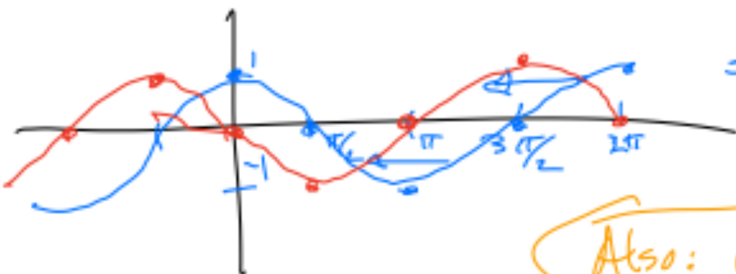
then the slopes of the cos(x) graph are the same as the slopes of the sin(x) graph, moved over $\frac{\pi}{2}$ to left.

$$(\cos(x))' = \sin'\left(x+\frac{\pi}{2}\right)$$

$= \cos(x)$ moved left $\pi/2$

$$= \cos\left(x+\frac{\pi}{2}\right)$$

$$= -\sin(x)$$




Also: $\cos\left(x+\frac{\pi}{2}\right) = \cos(x)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right)$

$= 0 - \sin(x)\cdot 1$

$= -\sin(x)$

$$\therefore (\cos(x))' = -\sin(x).$$

Yay!

Product Rule: $(f(x)g(x))'$

$$= f'(x)g(x) + f(x)g'(x)$$

(if $f$ & $g$ are differentiable at $x$.)

**Important Fact:** If a function $f$ is differentiable at a point $a \in \mathbb{R}$, then $f$ is continuous at $a$.

Idea of proof: $f'(a)$ exists $\Rightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

Since $h \to 0$ (denominator), the numerator must also go to zero. $\Rightarrow \lim_{h\to 0}(f(a+h)-f(a)) = 0 \Rightarrow \left(\lim_{h\to 0} f(a+h)\right) = f(a)$

$\Rightarrow \lim_{x\to a} f(x) = f(a)$. $\boxed{\checkmark}$

**Proof of Product Rule**

Let $P(x) = f(x)g(x)$

Consider $\frac{P(x+h)-P(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$$= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \frac{\left(f(x+h)-f(x)\right)g(x+h) + f(x)\left(g(x+h)-g(x)\right)}{h}$$

$$= \underbrace{\frac{f(x+h)-f(x)}{h}}_{\to f'(x)} \underbrace{g(x+h)}_{\to g(x)} + f(x)\underbrace{\frac{\left(g(x+h)-g(x)\right)}{h}}_{\to g'(x)}$$

since $g$ is diff'ble at $x$ $\Rightarrow$ continuous at $x$

$\therefore \lim_{x\to 0} \frac{P(x+h)-P(x)}{h} = f'(x)g(x) + f(x)g'(x)$. $\boxed{\checkmark}$

## Longer Product Rule (by using the previous product rule several times.)

$$\left(\left(f(x)g(x)\right)\left(q(x)r(x)\right)\right)'$$

$$= f'(x)g(x)q(x)r(x) + f(x)g'(x)q(x)r(x) + f(x)g(x)q'(x)r(x) + f(x)g(x)q(x)r'(x).$$

---

**Example:** Find the derivative of

$$y = \frac{\cos(x)\cdot e^x}{\sqrt{x}}$$

$$\frac{C}{D} = \frac{1}{D}\cdot C$$

$$\frac{1}{x^B} = \frac{x^{-B}}{1}$$

---

**Solution:** $y = x^{-1/2}\cdot\cos(x)\cdot e^x$

$$\Rightarrow \frac{dy}{dx} = \left(x^{-1/2}\right)'\cdot\cos(x)e^x + \left(x^{-1/2}\right)\left(\cos(x)\right)'e^x + \left(x^{-1/2}\right)\left(\cos(x)\right)\left(e^x\right)'$$

$$= \left(\frac{-1}{2}\right)x^{-3/2}\cdot\cos(x)e^x + x^{-1/2}\left(-\sin(x)\right)\cdot e^x + x^{-1/2}\cos(x)e^x$$

$$= \boxed{\frac{-1}{2}x^{-3/2}\cos(x)e^x - x^{-1/2}\sin(x)e^x + x^{-1/2}\cos(x)e^x}$$

<u>Quotient Rule</u> If $f$ & $g$ are differentiable at $x$ and $g(x) \neq 0$, then

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

$$\left(\frac{Hi}{Ho}\right)' = \frac{Ho\, dHi - Hi\, dHo}{Ho\, Ho}$$

---

Proof: Let $G(x) = \dfrac{f(x)}{g(x)}$.

$$\frac{G(x+h) - G(x)}{h} = \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \left(\frac{g(x)f(x+h)}{g(x)g(x+h)} - \frac{f(x)\,g(x+h)}{g(x)g(x+h)}\right)\frac{1}{h}$$

$$= \frac{g(x)f(x+h) - f(x)g(x+h)}{g(x)\;g(x+h)\cdot h}$$

$$= \frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - f(x)g(x+h)}{g(x)\;g(x+h)\;h}$$

$$= \frac{g(x)\left(f(x+h) - f(x)\right)}{g(x)\,g(x+h)\cdot h} + \frac{f(x)\left(g(x) - g(x+h)\right)}{g(x)\;g(x+h)\;h}$$

$$= \frac{g(x)}{g(x)\,g(x+h)} \underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)} - \frac{f(x)}{g(x)\,g(x+h)} \underbrace{\left(\frac{g(x+h)-g(x)}{h}\right)}_{g'(x)}$$

with $g(x+h) \to g(x)$.

$$\Rightarrow \lim_{h\to 0} \frac{G(x+h)-G(x)}{h} = \frac{g(x)}{(g(x))^2}\cdot f'(x) - \frac{f(x)}{(g(x))^2}\, g'(x)$$

$$= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}. \quad \blacksquare$$

---

Let's use this.

Example $(\tan(x))' = ?$

Solution: $(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)'$

$$= \frac{\text{Ho dHi} - \text{Hi dHo}}{\text{Ho Ho}} = \frac{\cos(x)\cdot(\sin(x))' - \sin(x)\cdot(\cos(x))'}{(\cos(x))^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}.$$

$$= \frac{1}{\cos^2(x)}.$$

$$\boxed{(\tan(x))' = \frac{1}{\cos^2(x)}} = \sec^2(x)$$

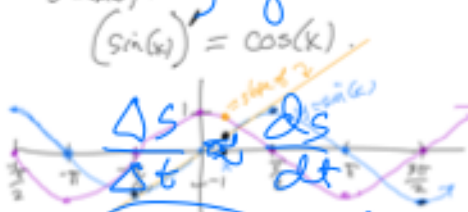
How do we find derivatives from data? (ie rates of change).

Example: Position of Isabella's car at different times.

| $t$ second | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s$ feet | 0 | 40 | 60 | 55 | 52 | 70 | 100 | 150 | 160 | 140 |

incr    decr    incr    dec.

Question: ① Estimate the velocity of Isabella's car at $t = 6$ sec.




$$\frac{\Delta s}{\Delta t} \qquad \frac{ds}{dt}$$

$$s'(6) \simeq \frac{100-70}{6-5} = \frac{30}{1} = 30 \,\frac{ft}{sec}.$$ Answer #1

$$s'(6) \approx \frac{150-100}{7-6} = \frac{50}{1} = 50 \,\frac{ft}{sec}$$

$$\text{Avg} = \frac{30+50}{2} = \boxed{40 \,\frac{ft}{sec}.}$$

$$s'(6) \approx \frac{150-70}{7-5} = \frac{80}{2} = \boxed{40 \text{ ft/sec}}$$